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Error estimates for data-driven optimal control by leveraging results for autonomous systems ★

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This work pursues the central task to efficiently solve *optimal control problems* for complex – and thus, expensive-to-evaluate – dynamical systems with the help of data-driven surrogate models. Mathematically speaking, we consider the following problem over the time horizon $p \cdot \Delta t$:

$$\begin{aligned} \min_{u \in U^p} J(y) &= \min_{u \in U^p} \sum_{i=0}^{p-1} P(y_{i+1}) \\ \text{s.t.} \quad y_{i+1} &= \Phi(y_i, u_i), \qquad i = 0, 1, 2, \ldots, \end{aligned} \tag{I}$$

where y_i and $u_i \in U$ are the system state and control at time instant $t_i = i\Delta t$. The objective function (for instance, the distance to some desired trajectory y^{ref}) is denoted by P, and Φ describes the flow of the underlying dynamical system (e.g., an ordinary or a partial differential equation) over the time increment Δt. The solution of (I) yields the optimal control u^* and corresponding state y^*.

A substantial challenge that we often face is the fact that the efficient prediction (and, by extension, control) of complex dynamical systems is hindered by the fact that the system dynamics are either very expensive to simulate or even unknown. Researchers have been investigating ways to accelerate the solution by using data for decades, the *Proper Orthogonal Decomposition (POD)* being an early and very prominent example (Sirovich, 1987). More recently, the major advances in data science and machine learning have lead to a plethora of new possibilities, for instance artificial neural networks, sparse regression for the identification of nonlinear dynamics (Brunton et al., 2016), or numerical approximations of the *Koopman operator* (Rowley et al., 2009; Klus et al., 2020), which describes the linear dynamics of observable functions. These methods facilitate the efficient simulation and prediction of high-dimensional spatio-temporal dynamics using measurement data, without requiring prior system knowledge. For control systems, a drawback is that the construction of surrogate models with inputs is often much more tedious and also problem-specific and data hungry (Bieker et al., 2020).

The approach we present here to solve (I) via surrogate models while avoiding the aforementioned issues is based on modifying the control problem instead of adjusting the surrogate modeling to the control setting. The resulting framework, which we call *QuaSiModO*, consists of the following steps (cf. also Figure 1):

(1) **Qua**ntization of the the admissible control U (for instance by replacing the interval $U = [u^{\mathsf{min}}, u^{\mathsf{max}}]$ by the bounds $V = \{u^{\mathsf{min}}, u^{\mathsf{max}}\}$);
(2) **Si**mulation of the autonomous systems with fixed inputs (e.g., $\Phi_{u^{\mathsf{min/max}}}(y) = \Phi(y, u^{\mathsf{min/max}})$);
(3) **Mod**eling of the individual systems via an arbitrary "off-the-shelf" surrogate modeling technique;
(4) **O**ptimization using the resulting set of autonomous surrogate models and relaxation techniques.

This interplay between continuous and integer control modeling as well as between the full system state and observed quantities (e.g., measurements) allows us to utilize the best of both worlds, namely

- integer controls for efficient data-driven modeling,
- continuous control inputs for real-time control, and
- existing error bounds for predictive models.

QuaSiModO successively transforms Problem (I) into related control problems that – as long as the predictive surrogate model is sufficiently accurate – yield optimal trajectories y^* that are close to one another. From (I) to (II), we quantize the control, meaning that only a finite set $V \subseteq U$ of inputs is feasible. This allows us to replace the non-autonomous dynamical system $\Phi(y, u)$ by a finite set of autonomous systems $\Phi_{u^j}(y)$, each corresponding

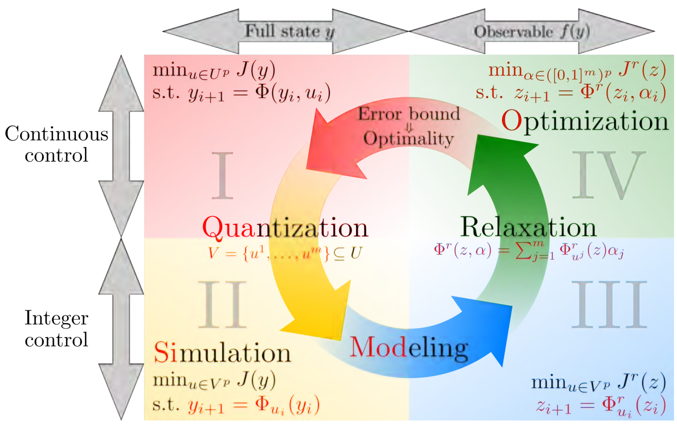

Fig. 1. The QuaSiModO framework consisting of the four steps Quantization, Simulation, Modeling and Optimization (Peitz and Bieker, 2021).

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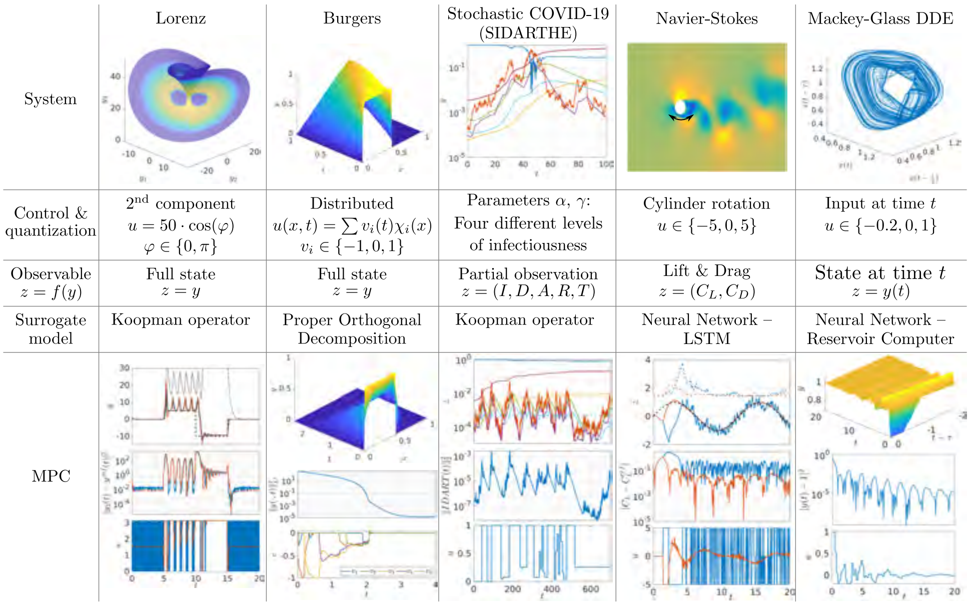

Fig. 2. QuaSiModO applied to various combinations of systems and surrogate models for model predictive control.

to one entry $u^j \in V$. While introducing an artificial drawback from the control perspective (Problem (II) is a mixed-integer optimal control problem), we can now easily introduce an equivalent Problem (III) that is based on surrogate models $\Phi^r_{u^i}(z)$ for a reduced quantity $z = f(y)$. Here, the function f is an *observable* which maps measurements from the state space of the full system to the space of measurements (which may be of significantly smaller dimension). As the transformation from (II) to (III) acts on a set of autonomous systems, we can approximate the individual systems Φ_{u^j} from individual measurement data sets, using whichever method we prefer.

In order to mitigate the disadvantages with respect to the complexity of the control problem, the problem of selecting an optimal input from V is relaxed by determining the optimal convex combination of the autonomous systems:

$$\begin{aligned} &\min_{\alpha \in ([0,1]^m)^p} J^r(z) = \min_{\alpha \in ([0,1]^m)^p} \sum_{i=0}^{p-1} P^r(z_{i+1}) \\ &\text{s.t.} \quad z_{i+1} = \sum_{j=1}^{m} \alpha_{i,j} \Phi^r_{u^j}(z_i), \qquad \sum_{j=1}^{m} \alpha_{i,j} = 1. \end{aligned} \tag{IV}$$

Problem (IV) is again continuous – with respect to the input α. For control affine systems, we can directly apply $u^* = \sum_{j=1}^m \alpha^*_j u^j$ to the real system. For non-affine systems, we use the sum up rounding algorithm from (Sager et al., 2012), by which a control corresponding to one of the quantized inputs is applied to the real system.

Besides the ability to include arbitrary models, an important aspect is that existing error bounds for the chosen surrogate model can easily be included, see (Peitz and Bieker, 2021) for a detailed description The availability of error bounds is of particular importance for engineering systems, where safety is of utmost importance (e.g., for aircraft or autonomous vehicles). The bounds guarantee the performance of a controller and – more importantly – will automatically become stronger with future developments in the field of data-driven modeling.

We have tested the QuaSiModO framework on a variety of dynamical systems, observable functions and surrogate modeling techniques, cf. Figure 2, a detailed description is given in (Peitz and Bieker, 2021). For instance, we can control the lift force acting on a cylinder (determined by the velocity and pressure fields governed by the 2D Navier–Stokes equations) without any knowledge of the flow field using the standard LSTM framework included in *TensorFlow*, and stabilize the Mackey-Glass equation using a standard echo state network. This highlights the flexibility and broad applicability of the method and the success of the technique in constructing data-driven feedback controllers.

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