Extended Abstract: Damage Modeling for the Tree-Like Network with Fractional-Order Calculus ⋆

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Introduction: This paper shows that a tree-like network with damage can be modeled as the product of a fractional-order (FO) nominal plant and a FO multiplicative disturbance, which is well structured and completely characterized by the damage amount at each damaged component. Such way of modeling brings insight about that damaged network's behavior and helps us design robust controllers under uncertain damages and identify the damage.

We study the network in Fig. 1, motivated by a viscoelastic model from Heymans and Bauwens (1994) and also studied in Goodwine (2014); Leyden (2018); Mayes (2012). Considering only integer-order calculus, that system can only be modeled by an infinite continued fraction. Existing literature, *e.g.*, Goodwine (2014), shows that, if FO calculus is allowed, then the undamaged version of that system is exactly half order which has a concise representation. This paper shows that for such a damaged network, its transfer function can still be written in a structured way.

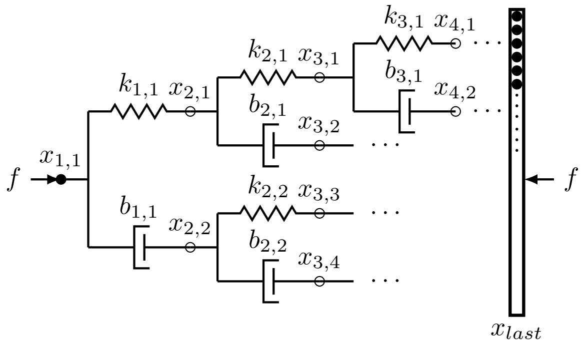

Fig. 1. The tree model.

It can be shown that the transfer function $\widetilde{G}(s)$ from the input force, f, to the distance between $x_{1,1}$ and x_{last} of such model satisfies the recurrence formula given by

$$\widetilde{G}(s) = \frac{1}{\dfrac{1}{\dfrac{1}{\widetilde{k}} + \widetilde{G}_U(s)} + \dfrac{1}{\dfrac{1}{\widetilde{b}s} + \widetilde{G}_L(s)}}. \tag{1}$$

Moving one generation deeper, the transfer function from the input force to the distance between $x_{2,1}$ and x_{last} is

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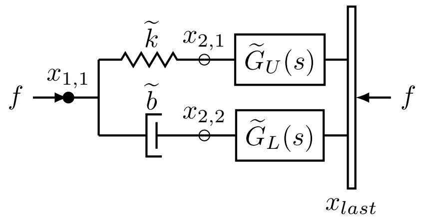

Fig. 2. An illustration about recurrence formula, Eq. (1).

$\widetilde{G}_U(s)$; similarly, $\widetilde{G}_L(s)$ is that between $x_{2,2}$ and x_{last}. The spring constant connecting $x_{1,1}$ to $x_{2,1}$ is denoted by $\widetilde{k}$, and $\widetilde{b}$ denotes the damper constant connecting $x_{1,1}$ to $x_{2,2}$. Fig. 2 illustrates the meaning of above elements.

We call the tree model *undamaged* when all spring and all damper constants are same, that is $k_{g,n} = k$ and $b_{g,n} = b$ for all $g = 1, 2, \ldots$ and $n = 1, 2, \ldots, 2^{g-1}$. For each damage case, we assume that there is either only one spring or only one damper having a constant different from its corresponding undamaged value. We further assume that the damaged spring (damper) constant k_d (b_d) is defined by a factor of ϵ, *i.e.*, $k_d = \epsilon k$ or $b_d = \epsilon b$, where ϵ is called the damage amount and $0 < \epsilon < 1$.

As shown in Goodwine (2014) and as is well-known, for the undamaged case, the transfer function from the input force $f(t)$ to the relative distance between $x_{1,1}(t)$ and $x_{\text{last}}(t)$ for the undamaged tree is given by

$$G_\infty(s) = \frac{X_{1,1}(s) - X_{\text{last}}(s)}{F(s)} = \frac{1}{\sqrt{kbs}}. \tag{2}$$

Eq. (1) can be viewed as a mapping from $(\widetilde{G}_U(s), \widetilde{G}_L(s))$ to $\widetilde{G}(s)$, which builds up the tree generation by generation regardless of whether the model is undamaged or damaged.

The existing literature outlined above shows that the undamaged tree's transfer function $G_\infty(s)$ in Eq. (2) can be obtained by replacing $(\widetilde{G}(s), \widetilde{G}_U(s), \widetilde{G}_L(s), \widetilde{k}, \widetilde{b})$ with $(G_\infty(s), G_\infty(s), G_\infty(s), k, b)$ in Eq. (1), *i.e.*, the undamaged transfer function between $x_{1,1}$ and x_{last} is the same as the one between $x_{2,1}$ and x_{last}, and also the one between $x_{2,2}$ and x_{last}.

In a similar manner and using self-similarity, every damage case can also be computed by using Eq. (1) repeatedly. However, repeatedly applying the above process will result

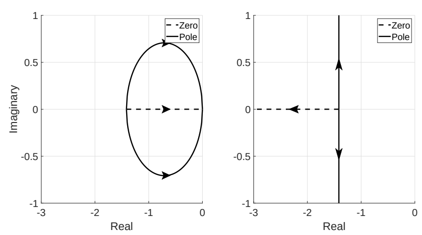

Fig. 3. Half-order ZP locus when the damage occurs at the first generation and $\epsilon \in [0, 1]$. When $\epsilon = 1$, all half-order ZP are at $-\sqrt{k/b}$. (For this plot, $k = 2$ and $b = 1$.) Left: $l = k_{1,1}$. Right: $l = b_{1,1}$.

in a very complicated transfer function. In fact, both transfer functions $\widetilde{G}_U(s)$ and $\widetilde{G}_L(s)$ have the same formula as Eq. (1) due to the self-similarity. Therefore, with integer-order calculus, the transfer function for the entire tree is a complicated infinite continued fraction.

Main Result: The main result of this work is that the damaged tree's transfer function can be written as

$$G_l(s) = G_\infty(s)\Delta_l(s),$$

where the disturbance $\Delta_l(s)$ is well structured and can be determined completely by the damage amount ϵ of a damaged component l. Those two features are the key points which make such way of modeling useful in different applications. See Ni (2021) for a complete analysis.

Claim: For each damage case outlined above, its damaged transfer function $G_l(s)$ from the input force to the relative distance between $x_{1,1}$ and x_{last} can be modeled as a FO nominal plant with a FO multiplicative disturbance,

$$G_l(s) = G_\infty(s)\Delta_l(s), \tag{3}$$

where $G_\infty(s)$ is the undamaged transfer function defined by Eq. (2). Moreover, $\Delta(s)$ is structured as

$$\Delta_l(s) = \frac{N(s)}{D(s)} = \frac{\prod_{j=1}^{2g}(s^{\frac{1}{2}} + z_j)}{\prod_{j=1}^{2g}(s^{\frac{1}{2}} + p_j)} \tag{4}$$

where g denotes the g-th generation at which the damaged component l locates, and $-z_j$ and $-p_j$ are called as half-order zeros and poles. In addition, z_1 is fixed at $\sqrt{k/b}$ regardless of the damage location or amount ϵ.

Claim: $\Delta_l(s)$ Depends on ϵ only at each l.

When the damage happens at the first generation, the relation between $\Delta_l(s)$ and ϵ can be expressed in closed-form. Fig. 3 shows the locus for those half-order zeros and poles when the damage happens at the first generation, and when the damage amount ϵ varies from 1 (no damage) to 0 (complete damage).

For all the other damage locations deeper into the network than the first generation, the relation between $\Delta_l(s)$ and ϵ cannot be easily expressed in a closed form. However,we can still obtain those locus by using a nonlinear equation solver. Fig. 4 shows the locus for those half-order zeros and poles, which are built up numerically, when the damage happens at the second generation with damage $\epsilon \in [0, 1]$.

Since it is possible to get this kind of locus for each damaged component, $\Delta_l(s)$ clearly has only one degree of freedom, namely ϵ, at each damaged component l. That is, as long as either one pole or one zero (other than $-z_1$ which always stays at $-\sqrt{k/b}$) is known, all the other zeros and poles can be determined through ϵ, thus $\Delta_l(s)$ is determined thereby.

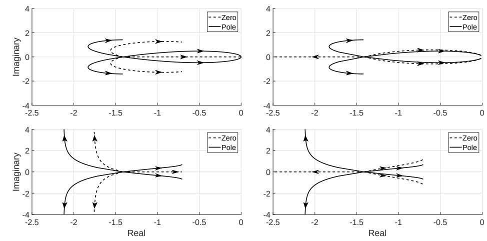

Fig. 4. Half-order ZP locus when the damage occurs at the second generation, and $\epsilon \in [1, 0]$. When $\epsilon = 1$, all half-order zeros and poles are at $-\sqrt{k/b}$. (For this plot, $k = 2$ and $b = 1$.) Upper left: $l = k_{2,1}$. Upper right: $l = k_{2,2}$. Lower left: $l = b_{2,1}$. Lower right: $l = b_{2,2}$.

Utility of These Results: Because the disturbance $\Delta_l(s)$ is completely determined by the damage amount ϵ of a damaged component l, we can use the above result to identify a damage tree network's damage amount ϵ. Specifically, we can formulate that damage identification problem as an optimization problem. For instance, when a damage occurs at $k_{2,1}$, given a frequency domain measurement $\Delta_{k_{2,1}}(s)$, we can identify its damage amount ϵ by solving the following optimization problem,

$$\min_{\epsilon} \sum \frac{\|\widetilde{\Delta}_{k_{2,1}}(s) - \Delta_{k_{2,1}}(s)\|}{\|\Delta_{k_{2,1}}(s)\|}$$

$$\text{where} \quad \widetilde{\Delta}_{k_{2,1}}(s) = \frac{\prod_{j=1}^{4}(s^{\frac{1}{2}} + z_j)}{\prod_{j=1}^{4}(s^{\frac{1}{2}} + p_j)}$$

and $z_j = z_j(\epsilon)$, $p_j = p_j(\epsilon)$ for all $j = 1, \ldots, 4$. The functions $z_j(\epsilon)$ and $p_j(\epsilon)$ are already known by fitting the ZP locus as shown in Fig. 4. We have successfully identified the damage amount ϵ by using `fmincon()` to solve the above optimization problem.

Applications:

(1) Providing insights about how damage affects the network.
(2) Robust control.
(3) Identification of samage for a damaged network.

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