An Alternative Algorithm for Unstable Balanced Truncation ⋆

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Abstract: Model reduction of stable linear-time invariant systems by balanced truncation is well-established in systems and control engineering. For unstable systems, several alternatives have been suggested, with linear-quadratic Gaussian balanced truncation arguably the most prominent one. Here, we discuss an alternative method that can be computed in a potentially more efficient way.

Keywords: Model Reduction, Balanced Truncation, Algebraic Riccati Equations, Numerical Methods, Sign Function Method.

1. INTRODUCTION

Balancing-related model order reduction (MOR) is one of the main techniques for reducing the complexity of linear dynamical systems

$$\dot{x}(t) = Ax(t) + Bu(t), \qquad y(t) = Cx(t), \tag{1}$$

and is particularly popular in systems and control engineering due to its beneficial properties for control system design (e.g. Antoulas (2005); Benner (2009); Baur et al. (2014); Benner et al. (2021)). The basic principle is to use two symmetric positive semidefinite matrices P, Q and a contragredient transformation to find a coordinate system in which they are equal and diagonal. Then one projects the dynamics of (1) onto the dominant subspace of $P = Q$ in this coordinate system. This is always possible if the system is controllable and observable, and can still be used on the controllable and observable subspaces for MOR purposes.

The most common choice (Moore (1981)) for P, Q is to use the system (reachability and obervability) Gramians which solve the two “dual” Lyapunov equations

$$AP + PA^T + BB^T = 0, \tag{2a}$$

$$A^TQ + QA + C^TC = 0. \tag{2b}$$

Note that for the usual SR or BFSR procedures to compute a reduced-order model from A, B, C and P, Q, one needs (approximations of) full-rank or Cholesky factors of P, Q, i.e. one works with S, R satisfying

$$P = SS^T, \qquad Q = R^TR$$

approximately. A prerequisite for this to be a successful is that A is stable, i.e., has all its eigenvalues in the open left half of the complex plane. The resulting method is commonly called *Balanced Truncation (BT).*

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2. BT FOR UNSTABLE SYSTEMS

One possibility to apply balancing-based MOR for unstable systems is to use *LQG balanced truncation (LQGBT)* (Jonckheere and Silverman (1983)). Here, $(P, Q) = (X_s, Y_s)$ is chosen, where X_s and Y_s are the unique stabilizing solutions of the algebraic Riccati equations (AREs) corresponding to the linear-quadratic regulator (LQR) and Kalman-Bucy filter problems related to (1):

$$A^TX + XA - XBB^TX + C^TC = 0, \tag{3a}$$

$$AY + YA^T - YC^TCY + BB^T = 0. \tag{3b}$$

An alternative to LQGBT is *closed-loop balancing* (Wortelboer (1994)). The idea is to first stabilize the system and then to use the Gramians of the closed-loop system in the balance-and-truncate procedure. Suppose one chooses for the stabilization X_s, the stabilizing solution of the LQR ARE (3a). This requires to first compute the unique stabilizing solution X_s of the LQR Riccati equation and then to apply the feedback law

$$u_s(t) = -B^TX_sx(t) + u(t)$$

to (1), resulting in the *closed-loop system*

$$\dot{x}_s(t) = (A - BB^TX_s)x_s(t) + Bu(t), \; y_s(t) = Cx_s(t). \tag{4}$$

Then, *closed-loop balanced truncation (CLBT)* uses the solutions P_s, Q_s of the Lyapunov equations

$$(A - BB^TX_s)P_s + P_s(A - BB^TX_s)^T + BB^T = 0, \tag{5a}$$

$$(A - BB^TX_s)^TQ_s + Q_s(A - BB^TX_s) + C^TC = 0. \tag{5b}$$

As it turns out, P_s can simply be computed by applying the sign function to the Hamiltonian matrix

$$\begin{bmatrix} A & -BB^T \\ -C^TC & -A^T \end{bmatrix} \tag{6}$$

associated to the LTI system (1), without ever computing X_s. In particular, P_s can be read off from $\mathrm{sign}(H)$ without further computation! This follows from the following corollary of the proof of (Kenney et al., 1989, Theorem 1).

Corollary 1. Let (A, B) be stabilizable, and (A, C) be detectable. Then the unique stabilizing solution X_s to the

ARE (3a) exists and is symmetric positive semidefinite. Hence, $A - BB^T X_s$ is stable, (5a) as well as (5b) have unique solutions $P_s = P_s^T \geq 0$, $Q_s = Q_s^T \geq 0$, resp., and it holds

$$\mathrm{sign}(H) = \begin{bmatrix} -I + 2P_s X_s & -2P_s \\ 2X_s P_s X_s - 2X_s & I - 2X_s P_s \end{bmatrix}. \qquad (7)$$

How to get Q_s solving (5b) is not so straightforward, though. It should be computed using any Lyapunov solver where one would then also need X_s to set up the coefficient matrix $A - BB^T X_s$.

It is interesting to note that the observability Gramian of yet another stabilized system can also be read off from sign(H). Here, one uses as "closed-loop matrix" $A - Y_s C^T C$, which is stable under the same assumptions as used in Corollary 1. The observability Gramian $\tilde{Q}_s$ of this stable LTI system solves the Lyapunov equation

$$(A - Y_s C^T C)^T \tilde{Q} + \tilde{Q}(A - Y_s C^T C) + C^T C = 0. \qquad (8)$$

Now, $\tilde{Q}_s$ can be obtained from the $(1,2)$-block of the sign function applied to the Hamiltonian matrix corresponding to (3b) which is nothing but H^T with H as in (6). As $\mathrm{sign}(H^T) = (\mathrm{sign}(H))^T$, we can read-off $\tilde{Q}_s$ from the $(2,1)$-block of sign(H). BT could now also be based on $(P, Q) = (P_s, \tilde{Q}_s)$, which to the best of our knowledge has not been described in the literature. As we will see from the numerical example below, this new balancing-based MOR method for unstable systems yields very good results, comparable to LQGBT.

2.1 Numerical Example

We use the eady data from the SLICOT benchmark collection[1]. Here, $n = 598$, $m = p = 1$. We computed reduced-order models (ROMs) of order $r = 17$ using BT and LQGBT as implemented in MORLAB (Benner and Werner (2020)). We also computed a ROM based on $(P_s, \tilde{Q}_s)$ as suggested above, where we used `signm` from MORLAB to compute sign(H) and read off the $(1,2)$- and $(2,1)$-blocks to get P_s and $\tilde{Q}_s$. We then obtained approximate full-rank factors of both matrices using truncated SVDs, and passed them to `srrom` from MORLAB to compute the reduced-order model, using the rank parameter set to $r = 17$. For now, we call this method "CLBT2". Fig. 1 shows the Bode magnitude plot for the full-order model and the three computed ROMs, where the graphs are indistinguishable in the "eyeball norm". The Bode magnitude plot of the errors for the three ROMs is displayed in Fig. 2. Here, the interesting fact arises that the error plots of LQGBT and CLBT2 coincide, and differ from that of BT. This supports the conjecture that CLBT2 and LQGBT actually compute the same ROM, i.e., that they are equivalent.

3. OUTLOOK

The discussion of the conjecture that the new "CLBT2" method is really just another (and potentially) more efficient implementation of LQBT will be part of the talk delivered at MATHMOD 2022. The proof of this conjecture will be reported elsewhere, as it requires more space than available here.

[1] `https://morwiki.mpi-magdeburg.mpg.de/morwiki/index.php/Earth_Atmosphere`

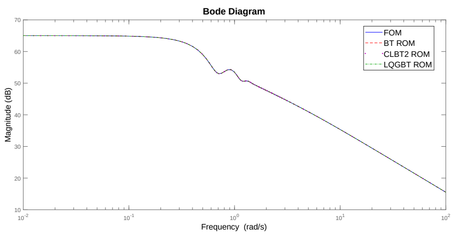

Fig. 1. Frequency responses for the full- and reduced-order models using the "eady data".

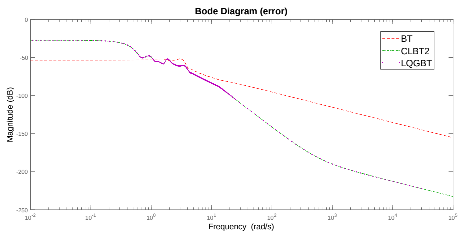

Fig. 2. Frequency responses for the errors of the reduced-order models using the "eady data".

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