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A Two-Dimensional Port-Hamiltonian Model for Coupled Heat Transfer

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Abstract: The problem of conjugate heat transfer in gas turbine blades and their cooling ducts is investigated by constructing a highly simplified mathematical model that focuses on the relevant coupling structures and aims to reduce the unrelated complexity as much as possible. Then, the Port-Hamiltonian formalism is applied to the model and its subsystems, and the interconnections are investigated. Finally, a simple spatial discretization is applied to the system to investigate the properties of the resulting finite-dimensional Port-Hamiltonian system and to determine whether the order of coupling and discretization has an effect on the resulting semi-discrete system.

Keywords: Port-Hamiltonian System, Conjugate Heat Transfer, Coupled System, Thermodynamics, heat equation, cooling channel

1. INTRODUCTION

In this discussion contribution we propose a simplified mathematical model of the coupled system of a heated blade and a cooling channel as it appears in modern gas turbines. First we develop a port-Hamiltonian system (PHS) formulation for each of the subsystems and investigate the coupling structure of their interconnection in order to determine whether the coupling of the two subsystems forms a PHS for the overall system. Next we propose some spatial discretization of the PHS and study whether the resulting semi-discrete systems form finite-dimensional PHS and whether there is a difference between the coupling of the discretized systems and the discretization of the coupled system. For details we refer to Jäschke et al. (2021).

2. THE MODEL SYSTEM

First, let us introduce the mathematical model of the coupled system under investigation. $\Omega_m = (0,1)\times(0,1) \subset \mathbb{R}^2$ denotes the spatial domain of the blade metal. The heat equation on Ω_m is given by

$$\frac{\partial T}{\partial t}(x,y,t) = \frac{1}{c_m\rho_m}\operatorname{div}\left(\lambda \operatorname{grad} T(x,y,t)\right), \; (x,y)\in\Omega_m. \tag{1}$$

In Figure 1 we give a rough sketch of the model setting.

The left, upper and lower boundary ($x=0$, $y=0$ and $y=1$), denoted by $\partial\Omega_{ext}$, are in contact with a thermal reservoir with a given temperature T_{ext}, leading to:

$$-\lambda\frac{\partial T}{\partial x}(x,y,t) = h_0\big(T_{\text{ext}}(t) - T(x,y,t)\big),\; x=0, y\in[0,1] \tag{2}$$

$$-\lambda\frac{\partial T}{\partial y}(x,y,t) = h_0\big(T_{\text{ext}}(t) - T(x,y,t)\big),\; x\in(0,1),\, y=0 \tag{3}$$

$$\lambda\frac{\partial T}{\partial y}(x,y,t) = h_0\big(T_{\text{ext}}(t) - T(x,y,t)\big),\; x\in(0,1), y=1 \tag{4}$$

The right boundary $\partial\Omega_c$ ($x=1$) is in contact with the cooling channel, so that $\partial\Omega_m = \partial\Omega_{\text{ext}} \cup \partial\Omega_c$ and

$$-\lambda\frac{\partial T}{\partial x}(1,y,t) = h_1\big(T(1,y,t) - \Theta(y,t)\big),\; (1,y)\in\partial\Omega, \tag{5}$$

with the temperature of the cooling channel Θ that is governed by a transport equation with an additional source term describing the heat flux into the cooling channel:

$$\frac{\partial\Theta}{\partial t}(y,t) = -v\frac{\partial\Theta}{\partial y}(y,t) + \frac{h_1}{c_c\rho_c}\big(T(1,y,t)-\Theta(y,t)\big), \tag{6}$$

$$\Theta(0,t) = \Theta_{\text{in}}(t). \tag{7}$$

3. PORT-HAMILTONIAN FORMULATION

We formulate the PHS for each subsystem using quadratic Hamiltonians. For the heat equation in the metal rod we

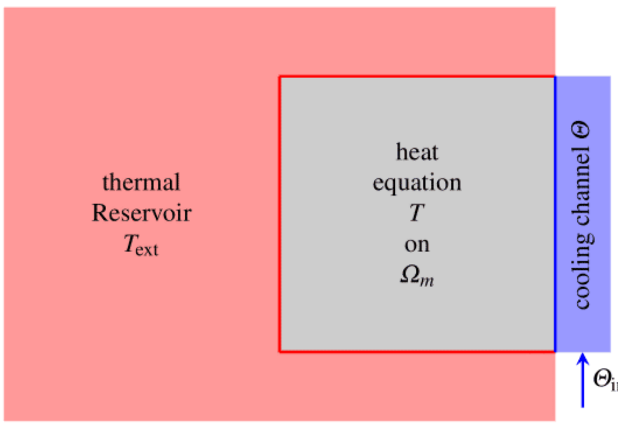

Fig. 1. Schematic of the 2D model system with $\partial\Omega_{\text{ext}}$ marked as a red line and $\partial\Omega_c$ as a blue line.

choose the Hamiltonian, cf. Serhani et al. (2019)

$$H(t) = \frac{1}{2}\int_{\Omega_m} \rho(\boldsymbol{x})c_m(\boldsymbol{x})\,T(t,\boldsymbol{x})^2\,\mathrm{d}\boldsymbol{x}, \tag{8}$$

with temperature $T(t,\boldsymbol{x})$ and c_m is the *isochoric specific heat capacity* that does not depend on the temperature.

We now choose the usual flow and effort variables

$$e_T = \delta_T H = T, \qquad f_T = \partial_t T, \tag{9}$$

with δ_T denoting the variational derivative w.r.t. T and the measure $\rho c_m\,\mathrm{d}x$.

Next, the first law of thermodynamics yields

$$\rho(\boldsymbol{x})c_m(\boldsymbol{x})\,\partial_t T(t,\boldsymbol{x}) = -\operatorname{div}\boldsymbol{\Phi}_Q(t,\boldsymbol{x}), \tag{10}$$

with the heat flux Φ_Q. The (isotropic) Fourier's law gives

$$\boldsymbol{\Phi}_Q(t,\boldsymbol{x}) = -\lambda\operatorname{grad}T(t,\boldsymbol{x}). \tag{11}$$

Therefore we introduce the additional flow and effort variables similar to Serhani et al. (2019)

$$\boldsymbol{e}_Q = \boldsymbol{\Phi}_Q, \quad \boldsymbol{f}_Q = -\operatorname{grad}T, \tag{12}$$

to obtain the system of equations

$$\begin{pmatrix}\rho c_m f_T\\ \boldsymbol{f}_Q\end{pmatrix} = \begin{pmatrix}0 & -\operatorname{div}\\ -\operatorname{grad} & 0\end{pmatrix}\begin{pmatrix}e_T\\ \boldsymbol{e}_Q\end{pmatrix}, \tag{13}$$

$$\boldsymbol{e}_Q = \lambda\boldsymbol{f}_Q. \tag{14}$$

The time derivative of the Hamiltonian reads

$$d_t H = -\int_{\Omega_m}\boldsymbol{e}_Q\boldsymbol{f}_Q\,\mathrm{d}x - \int_{\partial\Omega_m} e_T(\boldsymbol{e}_Q\boldsymbol{n})\,\mathrm{d}\gamma, \tag{15}$$

i.e. the same boundary port variables as Serhani et al. (2019). To replicate the boundary conditions, we set

$$\boldsymbol{e}_Q\boldsymbol{n} = \boldsymbol{\Phi}_Q\boldsymbol{n} = h_0\,(T - T_{\text{ext}})\text{ on }\partial\Omega_{ext} \tag{16}$$

so equation (15) becomes

$$\begin{aligned} d_t H = &-\int_{\Omega_m}\boldsymbol{e}_Q\boldsymbol{f}_Q\,\mathrm{d}x\\ &-\int_{\partial\Omega_{ext}} h_0 e_T^2\,\mathrm{d}\gamma + \int_{\partial\Omega_{ext}} h_0 e_T T_{\text{ext}}\,\mathrm{d}\gamma \end{aligned} \tag{17}$$

turning the boundary port of (15) into two new boundary ports and additional dissipative terms on the boundary.

For the treatment of the cooling channel we refer to Jäschke et al. (2021), providing us with a PHS that has an input $T(1,y,t)$ and an output $h_1\big(T(1,y,t) - \Theta(y,t)\big)$.

To obtain a PH formulation of the model system by coupling the PHS we need the following equality:

$$-\lambda\frac{\partial T}{\partial x}(1,y,t) = h_1\big(T(1,y,t) - \Theta(y,t)\big). \tag{18}$$

With the inputs and outputs of the two systems we find

$$e_1 = T(1,y,t), \quad f_1 = -\boldsymbol{\Phi}_Q\boldsymbol{n} = \lambda\frac{\partial T}{\partial x}(1,y,t), \tag{19}$$

$$e_2 = h_1\big(T(1,y,t) - \Theta(y,t)\big), \quad f_2 = T(1,y,t). \tag{20}$$

The 'gyrative' interconnection, cf. Cervera et al. (2007)

$$f_2 = e_1, \quad f_1 = -e_2, \tag{21}$$

is a Dirac structure, and obviously satisfies (18). Therefore, the combined system is again a port-Hamiltonian system.

4. DISCRETIZED COUPLED SYSTEMS

We employ a standard finite difference discretization to the PHS and due to its simplicity we can easily write down the matrices of the discretized system. Here, spatial grid variables are indicated by an underscore, e.g. $\underline{x}$.

We consider a uniform spatial grid with $N+1$ points and define $\underline{T} \in \mathbb{R}^{N\cdot M}$, such that $\underline{T}$ is defined on an offset grid, i.e. $\underline{T}_{i+jN} \approx T(\underline{x}_i + \frac{\Delta x}{2}, \underline{y}_j + \frac{\Delta y}{2})$. Meanwhile, the heat fluxes $\underline{\Phi_x} \in \mathbb{R}^{N\cdot M}$ and $\underline{\Phi_y} \in \mathbb{R}^{N(M+1)}$ are defined on a grid offset in only the y- and x-direction, respectively, i.e. $\underline{\Phi_x}_{i+Nj} \approx \Phi_x(\underline{x}_i, \underline{y}_j + \frac{\Delta y}{2})$ and $\underline{\Phi_y}_{i+Nj} \approx \Phi_y(\underline{x}_i + \frac{\Delta x}{2}, \underline{y}_j)$. We discretize the Hamiltonian (8) w.r.t. space using the midpoint rule

$$\underline{H} = \frac{1}{2}\rho c_m \Delta x\Delta y\,\underline{T}^\top\underline{T}, \tag{22}$$

giving us the internal energy change as flow variable and the temperature as effort variable:

$$\underline{f}^{(T)} = \rho c_m\Delta x\Delta y\,\frac{\partial\underline{T}}{\partial t}, \quad \underline{e}^{(T)} = \underline{T}. \tag{23}$$

The PHS modelling the cooling channel is disrectized similarly, cf. Jäschke et al. (2021).

Now the two PHS are coupled, results in a system with the Hamiltonian

$$H = \frac{1}{2}\int_\Omega \rho(\boldsymbol{x})c_m(\boldsymbol{x})\,T(t,\boldsymbol{x})^2\,\mathrm{d}\boldsymbol{x} + \frac{1}{2}\int_0^1 \rho_c c_c\,\Theta^2(y,t)\,\mathrm{d}y. \tag{24}$$

Discretizing T, Θ with a proper midpoint rule yields

$$\underline{H} = \frac{1}{2}\rho c_m\Delta x\Delta y\,\underline{T}^\top\underline{T} + \frac{1}{2}\Delta y\sum_{i=0}^{M-1}\rho_c c_c\underline{\Theta}_i^2, \tag{25}$$

the same Hamiltonian produced by coupling the two discretized PHS. We then obtain the following system:

$$\begin{pmatrix}f^{(T)}\\0\\0\\f^{(\Theta)}\end{pmatrix} = \begin{pmatrix}0 & J_x & J_y & 0\\ -J_x^\top & -R_x & 0 & B_{x,N}\\ -J_y^\top & 0 & -R_y & 0\\ 0 & -B_{x,N}^\top & 0 & J_\Theta - R_\Theta\end{pmatrix}\begin{pmatrix}e^{(T)}\\ \underline{\Phi_x}\\ \underline{\Phi_y}\\ e^{(\Theta)}\end{pmatrix} +$$

$$B\begin{pmatrix}T_{\text{ext}}(\underline{x}_0, \underline{y} + \dfrac{\Delta y}{2})\\ T_{\text{ext}}(\underline{x} + \dfrac{\Delta x}{2}, \underline{y}_0)\\ T_{\text{ext}}(\underline{x} + \dfrac{\Delta x}{2}, \underline{y}_M)\\ v\rho c_c\Theta_{in}\end{pmatrix}, \quad \underline{\tilde{w}} = B^\top\big(e^{(T)}, \underline{\Phi_x}, \underline{\Phi_y}, e^{(\Theta)}\big)^\top,$$

which is a PHDAE with J_Θ skew-symmetric and R_x, R_y, R_Θ symmetric.

At least in this case, for the discretization chosen here, there is therefore no difference between coupling the discretized systems and discretizing the coupled system.

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