Stabilization of the wave equation in port-Hamiltonian modelling ⋆

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Abstract: We investigate the stability of the wave equation with spatial dependent coefficients on a bounded multidimensional domain. The system is stabilized via a scattering passive feedback law. We formulate the wave equation in a port-Hamiltonian fashion and show that the system is semi-uniformly stable.

1. INTRODUCTION

In this paper we investigate the boundary control system

$$\begin{aligned}
u(t,\zeta) &= \frac{\partial w}{\partial T\nu}(t,\zeta), && \zeta\in\Gamma_1,\\
\frac{\partial^2 w}{\partial t^2}(t,\zeta) &= \frac{1}{\rho(\zeta)}\operatorname{div}\left(T(\zeta)\nabla w(t,\zeta)\right), && \zeta\in\Omega,\\
w(t,\zeta) &= h(\zeta), && \zeta\in\Gamma_0,\\
w(0,\zeta) &= w_0(\zeta), && \zeta\in\Omega,\\
\frac{\partial w}{\partial t}(0,\zeta) &= w_1(\zeta), && \zeta\in\Omega,\\
y(t,\zeta) &= \frac{\partial w}{\partial t}(t,\zeta), && \zeta\in\Gamma_1,
\end{aligned} \tag{1a}$$

with feedback law

$$u(t,\zeta) = -k(\zeta)y(t,\zeta), \qquad \zeta\in\Gamma_1, \tag{1b}$$

where $t\geq 0$, $\Omega\subseteq\mathbb{R}^n$ is a bounded domain with Lipschitz boundary $\partial\Omega=\overline{\Gamma_0}\cup\overline{\Gamma_1}$ with $\Gamma_0\cap\Gamma_1=\emptyset$, Γ_0 and Γ_1 are open in the relative topology of $\partial\Omega$ and the boundaries of Γ_0 and Γ_1 have surface measure zero. Furthermore, $w(\zeta,t)$ is the deflection at point $\zeta\in\Omega$ and $t\geq 0$, and profile h is given on Γ_0, where the wave is fixed. Let Young's elasticity modulus $T\colon\Omega\to\mathbb{C}^{n\times n}$ be a Lipschitz continuous matrix-valued function such that $T(\zeta)$ is a positive and invertible matrix (a.e.) and $T(\cdot)^{-1}\in\mathsf{L}^\infty(\Omega)^{n\times n}$. The vector ν denotes the outward normal at the boundary and $\frac{\partial}{\partial T\nu}w(t,\zeta)=T\nu\cdot\nabla w(t,\zeta)=\nu\cdot T\nabla w(t,\zeta)$ is the conormal derivative. The Lipschitz continuous mass density $\rho\colon\Omega\to\mathbb{R}_+$ satisfies $\rho,\frac{1}{\rho}\in\mathsf{L}^\infty(\Omega)$. Further, $k\colon\Gamma_1\to\mathbb{R}$ is a measurable positive and bounded function such that also its pointwise inverse is bounded, i.e. $k,\frac{1}{k}\in\mathsf{L}^\infty(\Gamma_1)$. Finally, w_0 and w_1 are the initial conditions.

Strong stability of (1) has been investigated in Quinn and Russell (1977). In Humaloja et al. (2019) this system also appears in port-Hamiltonian formulation, but with constant T and ρ and C^2 boundary. Under these restrictions, they show that this system is exponentially stable. However, semi-uniform stability, a notion which is stronger than strong stability and weaker than exponential stability, of (1) with spatial dependent functions ρ and T on quite general domains has not been studied so far.

We aim to show semi-uniform stability of (1) using a port-Hamiltonian formulation. Semi-uniform stability implies strong stability, and thus we extend the results obtained in Quinn and Russell (1977). To prove our main result we use the fact that semi-uniform stability is satisfied if the port-Hamiltonian operator generates a contraction semigroup and possesses no spectrum in the closed right half plane. Port-Hamiltonian systems encode the underlying physical principles such as conservation laws directly into the structure of the system structure. For finite-dimensional systems there is by now a well-established theory Maschke and van der Schaft (1992); Duindam et al. (2009). The port-Hamiltonian approach has been further extended to the infinite-dimensional situation, see e.g. Villegas (2007); Jacob and Zwart (2012); Kurula and Zwart (2015). In Kurula and Zwart (2015) the authors showed that the port-Hamiltonian formulation of the wave equation (1) possess unique mild and classical solutions.

2. PORT-HAMILTONIAN FORMULATION OF THE SYSTEM

We split the system (1) into a time independent system

$$\begin{aligned}
\operatorname{div}T(\zeta)\nabla w_{\mathrm{e}}(\zeta) &= 0, && \zeta\in\Omega,\\
w_{\mathrm{e}}(\zeta) &= h(\zeta), && \zeta\in\Gamma_0,\\
\frac{\partial w_{\mathrm{e}}}{\partial T\nu}(\zeta) &= 0, && \zeta\in\Gamma_1,
\end{aligned} \tag{2}$$

and a dynamical system

$$\begin{aligned}
\frac{\partial^2 w_{\mathrm{d}}}{\partial t^2}(t,\zeta) &= \frac{1}{\rho(\zeta)}\operatorname{div}(T(\zeta)\nabla w_{\mathrm{d}}(t,\zeta)), && \zeta\in\Omega,\\
w_{\mathrm{d}}(t,\zeta) &= 0, && \zeta\in\Gamma_0,\\
w_{\mathrm{d}}(0,\zeta) &= w_0(\zeta)-w_{\mathrm{e}}(\zeta), && \zeta\in\Omega,\\
\frac{\partial w_{\mathrm{d}}}{\partial t}(0,\zeta) &= w_1(\zeta), && \zeta\in\Omega,\\
\frac{\partial w_{\mathrm{d}}}{\partial T\nu}(t,\zeta) &= -k\frac{\partial w_{\mathrm{d}}}{\partial t}(t,\zeta), && \zeta\in\Gamma_1
\end{aligned} \tag{3}$$

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where $t \geq 0$. The original system is solved by $w(t,\zeta) = w_{\mathrm{e}}(t,\zeta) + w_{\mathrm{d}}(\zeta)$. As in Kurula and Zwart (2015) the system in (3) can be described in a port-Hamiltonian manner by choosing the state $x(t,\zeta) = \left[\begin{smallmatrix}\rho(\zeta)\frac{\partial}{\partial t}w_{\mathrm{d}}(t,\zeta)\\ \nabla w_{\mathrm{d}}(t,\zeta)\end{smallmatrix}\right]$. By using the convention

$$\begin{bmatrix}x_1(t)\\x_2(t)\end{bmatrix} := x(t) := x(t,\cdot)$$

we can write the system (3) as

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}x(t) &= \begin{bmatrix}0 & \operatorname{div}\\ \nabla & 0\end{bmatrix}\begin{bmatrix}\frac{1}{\rho} & 0\\ 0 & T\end{bmatrix}x(t),\\ x(0) &= \begin{bmatrix}\rho w_1\\ \nabla(w_0 - w_{\mathrm{e}})\end{bmatrix},\\ \gamma_0 \tfrac{1}{\rho}x_1(t)\big|_{\Gamma_0} &= 0,\\ \gamma_\nu T x_2(t)\big|_{\Gamma_1} &= -k\gamma_0\tfrac{1}{\rho}x_1(t)\big|_{\Gamma_1}\end{aligned}$$

By γ_0 and γ_ν we denote the boundary trace (extension of $f \mapsto f\big|_{\partial\Omega}$) and the normal trace (extension of $f \mapsto \nu \cdot f\big|_{\partial\Omega}$), respectively. Kurula and Zwart (2015) choose the state space $\mathsf{L}^2(\Omega)^{n+1}$ equipped with the energy inner product

$$\langle x, y\rangle := \left\langle x, \left[\begin{smallmatrix}\frac{1}{\rho} & 0\\ 0 & T\end{smallmatrix}\right] y\right\rangle_{\mathsf{L}^2(\Omega)^{n+1}},$$

which is equivalent to the standard inner product of $\mathsf{L}^2(\Omega)^{n+1}$ thanks to the assumptions on T and ρ. They then show the existence of mild and classical solution via semigroup methods. For well-posedness this is a suitable state space, but when it comes to stability this state space is too large as it does not reflect the fact that the second component of the state variable x_2 is of the form ∇v, for some function v in the Sobolev space $\mathsf{H}^1_{\Gamma_0}(\Omega)$. Thus, we choose the state space $\mathcal{X}_{\mathcal{H}}$ as $\mathsf{L}^2(\Omega) \times \nabla\mathsf{H}^1_{\Gamma_0}(\Omega)$, instead of $\mathsf{L}^2(\Omega)^{n+1}$. Note that $\nabla\mathsf{H}^1_{\Gamma_0}(\Omega)$ is closed in $\mathsf{L}^2(\Omega)^n$ by Poincaré's inequality. Hence, $\mathcal{X}_{\mathcal{H}}$ is also a Hilbert space with the L^2-inner product. Nevertheless, we also use the equivalent energy inner product on $\mathcal{X}_{\mathcal{H}}$, that is

$$\langle x, y\rangle_{\mathcal{X}_{\mathcal{H}}} := \left\langle x, \left[\begin{smallmatrix}\frac{1}{\rho} & 0\\ 0 & T\end{smallmatrix}\right] y\right\rangle_{\mathsf{L}^2(\Omega)^{n+1}}.$$

Furthermore, we define

$$\begin{aligned}\mathfrak{A} &:= \left[\begin{smallmatrix}0 & \operatorname{div}\\ \nabla & 0\end{smallmatrix}\right]\left[\begin{smallmatrix}\frac{1}{\rho} & 0\\ 0 & T\end{smallmatrix}\right]\\ \text{with}\quad \mathcal{D}(\mathfrak{A}) &:= \left[\begin{smallmatrix}\frac{1}{\rho} & 0\\ 0 & T\end{smallmatrix}\right]^{-1}\left(\mathsf{H}^1_{\Gamma_0}(\Omega) \times \mathsf{H}(\operatorname{div},\Omega)\right)\end{aligned}$$

as densely defined operator on $\mathsf{L}^2(\Omega)^{n+1}$. Note that we have already packed the boundary condition $\gamma_0\frac{1}{\rho}x_1 = 0$ on Γ_0 into the domain of $\mathfrak{A}$. Moreover, by construction $\operatorname{ran}\mathfrak{A} = \mathcal{X}_{\mathcal{H}}$. Taking the state space and the remaining boundary conditions (feedback) into account gives

$$\begin{aligned}A &:= \mathfrak{A}\big|_{\mathcal{D}(A)}, \qquad \text{where}\\ \mathcal{D}(A) &:= \left\{x \in \mathcal{D}(\mathfrak{A}) \;\middle|\; \gamma_\nu T x_2 = -k\gamma_0\tfrac{1}{\rho}x_1 \text{ on } \Gamma_1\right\} \cap \mathcal{X}_{\mathcal{H}}\end{aligned} \tag{4}$$

as an operator on $\mathcal{X}_{\mathcal{H}}$.

Proposition 1. The operator A given by (4) is a generator of contraction semigroup.

3. STABILITY RESULTS

Definition 2. We say a strongly continuous semigroup $(T(t))_{t\geq 0}$ on a Hilbert space X is *strongly stable*, if for every $x \in X$ we have $\lim_{t\to\infty}\|T(t)x\|_X = 0$.

We say a continuous semigroup $(T(t))_{t\geq 0}$ on a Hilbert space X is *semi-uniformly stable*, if there exists a continuous monotone decreasing function $f\colon [0,\infty) \to [0,\infty)$ with $\lim_{t\to\infty} f(t) = 0$ and

$$\|T(t)x\|_X \leq f(t)\|x\|_{\mathcal{D}(A)}, \qquad x \in \mathcal{D}(A).$$

Note that semi-uniform stability is also defined by $\|T(t)A^{-1}\| \to 0$ or $\|T(t)(1+A)^{-k}\| \to 0$ as in Batty and Duyckaerts (2008). However, this is equivalent to our definition. Semi-uniform stability implies strong stability.

We denote by A the operator given by (4) which is associated to the port-Hamiltonian formulation of (1). Our main result is the following theorem.

Theorem 3. The semigroup generated by A is semi-uniformly stable.

For the original system (1) strong stability of A translates to: There is a $w_{\mathrm{e}} \in \mathsf{H}^1(\Omega)$ such that for every initial values $w_0 \in \mathsf{H}^1(\Omega)$, $w_1 \in \mathsf{L}^2(\Omega)$ the solution w satisfies

$$\lim_{t\to\infty}\|w(t,\cdot) - w_{\mathrm{e}}(\cdot)\|_{\mathsf{H}^1(\Omega)} = 0.$$

4. CONCLUSION

In this paper we showed semi-uniform stability of the multidimensional wave equation equipped with a scattering passive feedback law.

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